

Resolving Gravitational Singularities

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ABSTRACT. We review recent progress on the resolution of gravitational singularities in string theory. The main example is the fundamental string in five dimensions which is singular in the standard supergravity description but regular after taking into account higher derivative corrections determined by anomalies and supersymmetry. The application of the AdS/CFT correspondence to geometries that require higher derivatives for regularity poses interesting challenges.

1. Introduction

In the last several years there has been significant progress on understanding higher derivative corrections to black holes in string theory [1, 2, 3, 4, 5, 6]. An important aspect of this development is that some interesting examples have been identified where gravitational singularities are resolved by higher derivative corrections to the action [7, 8, 9, 10, 11, 12, 13]. The prototypical example is a regular black hole that would have vanishing horizon area were it not for the higher derivative corrections in the Lagrangian. The resolution of gravitational singularities is an important problem in many different contexts so it is worthwhile discussing this aspect of higher derivative corrections separately from perturbative corrections to solutions that are already regular in the leading approximation. That is the purpose of this lecture.

The work discussed here is collected from a series of articles by Castro, Davis, Kraus, and the current author [14, 15, 16]. These articles develop the subject in the context of five dimensional supergravity corrected by the mixed gauge-gravitational Chern-Simons term:

$$(1.1) \quad \mathcal{L}_{\text{CS}} = \frac{1}{24 \cdot 16\pi^2} c_{2I} A^I \wedge \text{Tr } R \wedge R ,$$

and terms related to this by supersymmetry [17]. In this lecture we sacrifice generality and technical detail in order to keep the text brief and, hopefully, accessible also for interested non-experts. A much more comprehensive review appeared recently in [18].

Before considering any specifics, it is useful to recall some of the general challenges intrinsic to any attempt at resolving singularities using higher derivative terms:

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- *Exact symmetries* of the theory must be preserved by the higher derivative corrections. In our context, the Chern-Simons term (1.1) is required for anomaly cancellation. Furthermore, we go through much effort to preserve supersymmetry.
- *Effective field theory* demands that *all* terms of a given order should be taken into account. We address this by appealing to the uniqueness of the supersymmetric completion of the mixed gauge-gravitational Chern-Simons term (1.1).
- *Field redefinition ambiguities*, such as $g_{\mu\nu} \rightarrow g_{\mu\nu} + AR_{\mu\nu}$ (and generalizations involving matter fields), mix terms of different derivative order so that the singularity could depend on the choice of coordinates in field space. Our construction employs an off-shell formalism that implements supersymmetry at each order in the derivative approximation independently. This precludes most field redefinitions and defines the metric unambiguously.
- *Singularities* makes any systematic expansion inherently problematic. In order to cancel leading order singularities, the higher order terms must be comparable to the leading order terms. Then terms that are formally of even higher order might be important as well, *i.e.* there is no systematic expansion parameter. We will not be able to address this point completely but we will make it very explicit and concrete in due course.

We now turn to the case of small five-dimensional black strings in $N = 2$ supergravity, the example that is the focus of this talk.

2. Singularities or no singularities?

Consider M-theory on $CY_3 \times R^{4,1}$ for small CY_3 volume. The theory is effectively five dimensional and in the supergravity approximation it reduces to $N = 2$ supergravity in $D = 5$ coupled to a number of vector multiplets. The Lagrangian is

$$(2.1) \quad \mathcal{L}_0 = R + G_{IJ} \nabla M^I \nabla M^J + \frac{1}{2} G_{IJ} F_{ab}^I F^{Jab} - \frac{1}{24e} c_{IJK} A_a^I F_{bc}^J F_{de}^K \epsilon^{abcde} ,$$

in a self-explanatory notation (the theory is introduced in more detail in *e.g.* [19]).

We seek string solutions to this theory and so consider the *ansatz*:

$$ds^2 = e^{2U_1(r)} (dt^2 - dy^2) - e^{-4U_2(r)} (dr^2 + r^2 d\Omega_2^2) ,$$

for the geometry. The strings are supported by magnetic charges p^I with respect to the vector fields A^I , *i.e.* the two forms F^I

$$(2.2) \quad F^I = -\frac{1}{2} p^I e^{\hat{\theta}} \wedge e^{\hat{\phi}} ,$$

give rise to magnetic flux through the transverse two-sphere.

The warp factors $U_{1,2}(r)$ are going to diverge at $r = 0$ so there will at least be a coordinate singularity near the string. However, the singularity may just be a coordinate artifact, the actual geometry could be regular. This happens when $e^{-6U_1} = e^{-6U_2} \sim r^{-3}$ as $r \rightarrow 0$. Moreover, the resulting regular solutions have $AdS_3 \times S^2$ near string geometry.

The actual behavior of the string solution depends on the charge configuration. An important special case is when the matter supporting the string solution is the two-form $B_{\mu\nu}$ (represented by the one-form gauge-field that it dualizes to in

five dimensions) and the dilaton Φ , as in perturbative string theory. In this case the solution to the supergravity equations of motion has $e^{-6U_1} = e^{-6U_2} \sim r^{-1}$ as $r \rightarrow 0$. This solution is therefore singular in the supergravity approximation. We want to establish that higher derivative corrections to the theory can modify the warp factors near the string so that in fact $e^{-6U_1} = e^{-6U_2} \sim r^{-3}$ as $r \rightarrow 0$, leaving the full geometry regular.

For the class of string solutions considered here there is a simple and general criterion determining whether the geometry has a singularity. If the charge configuration is such that $c_{IJK}p^I p^J p^K$ is nonvanishing, then there are magnetic string solutions to standard (two-derivative) supergravity with the regular $AdS_3 \times S^2$ near string geometry with scale set by $c_{IJK}p^I p^J p^K$. However, if the charges are such that $c_{IJK}p^I p^J p^K = 0$, then the classical geometry is singular.

The string solutions in five dimensions are interpreted in M-theory as $M5$ -branes wrapped on four-cycles P in CY_3 . The magnetic charge p^I is the wrapping number around the basis four-cycle P_I . An $M5$ -brane wrapping a general four-cycle P , one not coinciding with any basis-cycle P_I , carries several (or many) of the charges p^I . The important combination $c_{IJK}p^I p^J p^K$ can be interpreted as the self-intersection number of the four-cycle P underlying the string solution, because the coupling constants c_{IJK} in the action (2.1) are interpreted microscopically as the intersection numbers of the four-cycles P_I .

We will be particularly interested in the special case where the CY_3 is the product manifold $K3 \times T^2$ and the $M5$ -brane wraps the four-cycle $P = K3$. This solitonic string is important because it is the type IIA dual of the heterotic string. Since the four-cycle underlying the dual heterotic string corresponds to one of the basis cycles it has vanishing self-intersection number $c_{IJK}p^I p^J p^K = 0$. The criterion using the self-intersection number $c_{IJK}p^I p^J p^K = 0$ thus reproduces the result that heterotic strings have singular near string geometry in the supergravity approximation.

3. Indirect Resolution of Singularities: Anomalies

The magnetic string solutions we consider are subject to powerful symmetry principles. These principles suggest when we should expect that gravitational singularities are resolved.

3.1. AdS/CFT correspondence. We must first recall the significance of the $AdS_3 \times S^2$ geometry. The global isometry group of AdS_3 is:

$$SO(2,2) \simeq SL(2) \times SL(2) .$$

This isometry induces an obvious global $SL(2) \times SL(2)$ symmetry acting on the asymptotic boundary of AdS_3 . The important point is that there is much more symmetry[20]: bulk diffeomorphisms combine with the global $SL(2) \times SL(2)$ and enhance the full boundary symmetry to $Vir \times Vir$. Similarly, the $SO(3) \simeq SU(2)$ isometry group of the S^2 combines with bulk diffeomorphisms and form the affine current algebra $\widehat{SU(2)}$ acting on the left-movers (the supersymmetric side) in the two-dimensional boundary CFT.

The bulk spacetime supersymmetry complements the bosonic isometries so that there is in fact a superisometry-group. Combining this with bulk diffeomorphisms

one finds a superconformal algebra on the boundary. In the case of $N = 2$ supersymmetry in bulk the correct superisometry is $SU(1, 1|2)$ and the boundary theory becomes a $(4, 0)$ superconformal CFT.

Explicit computation from the standard (two-derivative) supergravity action (2.1) determines the spacetime central charges of the Virasoro algebras[20]:

$$(3.1) \quad c_L = c_R = \frac{3\ell_A}{2G_3} ,$$

where ℓ_A is the length scale of the AdS_3 space. The central charge is a measure of the number of degrees of freedom in the two-dimensional boundary CFT. In bulk it is essentially the size of AdS_3 , suggesting that we can use central charge as a convenient proxy for the scale of the geometry.

The affine current algebra $\widehat{SU(2)}$ is similarly found to have level $k = c_L/6$, consistent with the $(4, 0)$ superalgebra. Geometrically, this relates the scale of the S^2 to that of AdS_3 . We will later see that the precise relation is $\ell_A = 2\ell_S$.

Expressing ℓ_A in terms of the magnetic charges p^I of the string, it can be shown that the supergravity central charge (3.1) can be written as:

$$(3.2) \quad c_L = c_R = c_{IJK} p^I p^J p^K .$$

In other words, in the supergravity approximation the scale ℓ_A is essentially the intersection number $c_{IJK} p^I p^J p^K$, which in turn is the central charge.

We previously noted that the vanishing (or not) of the intersection number $c_{IJK} p^I p^J p^K$ gives a criterion for whether the solution is singular (or regular). We now see that how this fits with the relation between central charges and the size of the AdS_3 (and of S^2). For charge configurations such that the supergravity central charge (3.2) vanishes, the corresponding scale vanishes and the geometry is singular.

We can now explain why we should expect that singularities are resolved: even when the classical central charge (computed in the supergravity approximation) vanishes, it is reasonable to expect that the exact central charge (including corrections) does not vanish. This motivates the expectation that the geometry should be regular when higher derivative corrections are taken into account.

3.2. Local Anomalies. The most robust higher derivative corrections are those related to anomaly cancellation. The standard Green-Schwarz terms (or alternatively terms inferred from M5-brane anomaly cancellation) give rise in five dimensions to the mixed gauge-gravitational Chern-Simons term

$$(3.3) \quad \begin{aligned} \mathcal{L}_{CS} &= -\frac{c_{2I}}{96e} A^I \wedge \text{Tr} R \wedge R \\ &= \frac{c_{2I}}{96e} F^I \wedge \omega_3 + \text{tot.der.} \end{aligned}$$

In the first line the interaction was written in a form that violates gauge symmetry, albeit only by a total derivative. The second line introduces the Chern-Simons three-form ω_3 (through $\text{Tr} R \wedge R = d\omega_3$) to make the term manifestly gauge invariant but then diffeomorphism symmetry is violated, albeit again just by a total derivative.

The AdS/CFT correspondence relates the diffeomorphism violations in bulk to corresponding anomalies in the boundary theory [12, 13]. The relation determines

the boundary central charges as:

$$(3.4) \quad c_L = c_{IJK} p^I p^J p^K + \frac{1}{2} c_{2I} p^I, \quad c_R = c_{IJK} p^I p^J p^K + c_{2I} p^I,$$

in accordance with other arguments [21, 22]. Both central charges are determined because diffeomorphism symmetry in AdS_3 and in S^2 (which is R-symmetry in the dual theory) give two pieces of information. The expressions (3.4) for the central charges are exact because the local symmetries that they enforce are exact. Even though supersymmetry plays an important role in the reasoning (relating R -charge and central charge on the supersymmetric side) we ultimately determine c_L (the supersymmetric side) and c_R with equal precision.

3.3. Resolution of singularities. The exact central charges (3.4) suggests the resolution of gravitational singularities. For this we consider charge vectors such that the intersection number $c_{IJK} p^I p^J p^K = 0$, corresponding to singular geometry in the leading order description. In this case the central charges (3.4) are linear in the magnetic charges

$$(3.5) \quad c_L = \frac{1}{2} c_{2I} p^I, \quad c_R = c_{2I} p^I,$$

and in general they do not vanish. To the extent that central charges can be taken as a measure of the scales of the near horizon $AdS_3 \times S^2$, we see that higher derivative corrections have replaced a singularity (a space with vanishing size) with a regular geometry (with nonvanishing scale). Thus a gravitational singularity has been resolved.

It is of special interest to consider the case of the dual heterotic string. Recall this is the case where the compact Calabi-Yau three fold is $CY_3 = K3 \times T^2$ and the $M5$ -brane simply wraps the four-cycle $P = K3$. Since $c_2(K3) = 24$ the central charges (3.5) become

$$c_L = 12p, \quad c_R = 24p.$$

These are the correct values for p heterotic strings in a physical gauge. (The left movers are 8 bosons and 8 fermions with central charge $c_L = 8 \cdot 1 + 8 \cdot \frac{1}{2} = 12$, the right movers are 24 bosons with central charge $c_R = 24 \cdot 1 = 24$). This gives confidence that both the central charges (3.5) have been correctly determined. It also suggests that the gravitational singularity has been resolved in the specific case of the heterotic string.

4. Explicit Singularity Resolution

So far we have used symmetries to argue that certain gravitational singularities are resolved. However, there are several ways the argument could fail. For example, we *assume* a $AdS_3 \times S^2$ near string geometry and then anomalies and symmetries determine the central charges. This reasoning would fail if the near string geometry was not $AdS_3 \times S^2$. Another point is that we use central charges as a proxy for the scale of the geometry and this relation is not precise when higher derivative corrections are taken into account. In order to understand these and other issues better we would like to explicitly construct asymptotically flat solutions with the anticipated properties. This is what we turn to next.

4.1. Supersymmetry. As we have already mentioned repeatedly, the essential ingredient is the mixed gauge-gravitational Chern-Simons term (3.3). Since the indirect arguments rely in part on supersymmetry, all terms related to the Chern-Simons term by supersymmetry must also be taken into account.

The direct way to supersymmetrize (3.3) would be to carry out the Noether procedure: act by the standard supersymmetry transformations and identify other terms in the Lagrangian. Then modify the supersymmetry transformations as needed and improve the supersymmetrization iteratively. The problem is that this procedure generates terms of arbitrarily high order so that there is no closed form of the Lagrangian.

This challenge can be addressed by using off-shell supermultiplets identified using the superconformal formalism. In the gauge fixed version of the superconformal formalism that we employ, the only complication relative to the on-shell formalism is that the standard physical fields must be augmented by auxiliary fields. In particular, the Weyl multiplet (*i.e.* gravity) has an auxiliary scalar D , auxiliary two-tensor v_{ab} , and also an auxiliary fermion χ .

The off-shell action is invariant under the supersymmetry transformations:

$$\begin{aligned}
 \delta\psi_\mu &= \left(\mathcal{D}_\mu + \frac{1}{2}v^{ab}\gamma_{\mu ab} - \frac{1}{3}\gamma_\mu\gamma\cdot v \right) \epsilon , \\
 \delta\Omega^I &= \left(-\frac{1}{4}\gamma\cdot F^I - \frac{1}{2}\gamma^a\partial_a M^I - \frac{1}{3}M^I\gamma\cdot v \right) \epsilon , \\
 \delta\chi &= \left(D - 2\gamma^c\gamma^{ab}\mathcal{D}_a v_{bc} - 2\gamma^a\epsilon_{abcde}v^{bc}v^{de} + \frac{4}{3}(\gamma\cdot v)^2 \right) \epsilon .
 \end{aligned}
 \tag{4.1}$$

It is important to emphasize that these transformations are symmetries of each order in the action by itself. Results we find by analyzing supersymmetry will therefore apply to the leading order solution, and also to the solution corrected by four derivative terms. Indeed, they must apply to the exact solution which take into account corrections to all orders.

The supersymmetry variations (4.1) take a form similar to the standard on-shell supersymmetries, except for the unfamiliar appearance of the auxiliary fields. We exploit these variations in the familiar way: demanding that the supersymmetry transformations vanish, when evaluated on purely bosonic backgrounds, yields linear differential equations satisfied by BPS-solutions.

As advertized previously, we assume as an *ansatz* that the metric takes the string form:

$$ds^2 = e^{2U_1}(dt^2 - dy^2) - e^{-4U_2}(dr^2 + r^2 d\Omega_2^2) .$$

Now the BPS conditions (4.1) impose $U_1 = U_2$ (so we can omit the index 1, 2 on U) and also determine the auxiliary fields in terms of the metric function U :

$$\begin{aligned}
 v &= \frac{3}{4}e^{2U}\partial_r U e^{\hat{\theta}} \wedge e^{\hat{\phi}} , \\
 D &= 6e^{4U}\nabla^2 U ,
 \end{aligned}
 \tag{4.2}$$

where ∇^2 is the Laplacian on the three-dimensional space transverse to the string. Finally, the BPS condition relates the magnetic fields and the scalar fields:

$$F^I = \frac{1}{2}\partial_r (M^I e^{-2U}) e^{4U} e^{\hat{\theta}} \wedge e^{\hat{\phi}} . \tag{4.3}$$

This is the standard attractor flow which represents field strengths as gradient flows, with the scalar fields essentially acting as potentials.

4.2. The Bianchi identity. The scalar fields M^I and the metric function U are not determined by supersymmetry alone. Generally, we must at this point appeal to the equations of motion which depend on the action. However, the case of 5D strings is special because it is magnetic fields

$$F^I = -\frac{1}{2}p^I e^{\hat{\theta}} \wedge e^{\hat{\phi}} ,$$

that support the string solution. These fields are topological and so they must be exact. However, the field strengths (4.3) determined from supersymmetry do not automatically take the correct form; they may even fail to satisfy the Bianchi identity. Imposing the Bianchi identity we find the harmonic equation

$$\nabla^2 (e^{-2U} M^I) = 0 .$$

In the spherically symmetric case assumed here the solution is

$$M^I e^{-2U} = H^I = M_\infty^I + \frac{p^I}{2r} ,$$

where M_∞^I are integration constants. These are the harmonic functions which underlie many familiar two derivative solutions. Since we have not yet specified the action, these are in fact *precisely* the standard harmonic functions, with normalizations and other details unchanged from the two derivative case.

4.3. The equations of motion. Up to this point we used supersymmetry and the Bianchi identity to determine the magnetic string solution completely, except for the metric factor $U(r)$. It is this function that depends on the detailed action. In order to get some familiarity with the off-shell formalism we first introduce the two-derivative action

$$(4.4) \quad \begin{aligned} \mathcal{L}_0 = & -\frac{1}{2}D + \frac{3}{4}\mathcal{R} + v^2 + \mathcal{N} \left(\frac{1}{2}D + \frac{1}{4}\mathcal{R} + 3v^2 \right) + 2\mathcal{N}_I v^{ab} F_{ab}^I \\ & + \mathcal{N}_{IJ} \left(\frac{1}{4}F_{ab}^I F^{Jab} + \frac{1}{2}\nabla M^I \nabla M^J \right) + \frac{1}{24e} c_{IJK} A_a^I F_{bc}^J F_{de}^K \epsilon^{abcde} . \end{aligned}$$

This action appears much more complicated than the standard supergravity action (2.1). However, the auxiliary fields D and v_{ab} enter algebraically so they can be integrated out exactly, by imposing their equation of motion. The equations of motion for D gives

$$(4.5) \quad \mathcal{N} = \frac{1}{6} c_{IJK} M^I M^J M^K = 1 .$$

This is the familiar special geometry constraint. It ensures that the number of independent physical scalar fields is one less than the number of vector fields, as must be the case because one of the vector fields is the graviphoton which has no scalar superpartner. Simplifying the Lagrangian (4.4) using the special geometry constraint (4.5) and the equation of motion for v_{ab} we recover the familiar on-shell $N = 2$ Lagrangian (2.1) with the correct expression

$$G_{IJ} = \frac{1}{2} (\mathcal{N}_I \mathcal{N}_J - \mathcal{N}_{IJ}) ,$$

for the metric on moduli space.

We are now ready to introduce the four derivative action in all its glory [17]

$$\begin{aligned}
(4.6) \quad \mathcal{L}_1 = & \frac{c_2 I}{24} \left(\frac{1}{16} \epsilon_{abcde} A^{Ia} R^{bcfg} R_{fg}^{de} + \frac{1}{8} M^I C^{abcd} C_{abcd} + \frac{1}{12} M^I D^2 + \frac{1}{6} F^{Iab} v_{ab} D \right. \\
& + \frac{1}{3} M^I C_{abcd} v^{ab} v^{cd} + \frac{1}{2} F^{Iab} C_{abcd} v^{cd} + \frac{8}{3} M^I v_{ab} \mathcal{D}^b \mathcal{D}_c v^{ac} - \frac{16}{9} M^I v^{ac} v_{cb} R_a^b \\
& - \frac{2}{9} M^I v^2 R + \frac{4}{3} M^I \mathcal{D}^a v^{bc} \mathcal{D}_a v_{bc} + \frac{4}{3} M^I \mathcal{D}^a v^{bc} \mathcal{D}_b v_{ca} - \frac{2}{3} M^I \epsilon_{abcde} v^{ab} v^{cd} \mathcal{D}_f v^{ef} \\
& + \frac{2}{3} F^{Iab} \epsilon_{abcde} v^{cf} \mathcal{D}_f v^{de} + F^{Iab} \epsilon_{abcde} v^c \mathcal{D}_f v^{ef} - \frac{4}{3} F^{Iab} v_{ac} v^{cd} v_{db} \\
& \left. - \frac{1}{3} F^{Iab} v_{ab} v^2 + 4 M^I v_{ab} v^{bc} v_{cd} v^{da} - M^I (v^2)^2 \right) ,
\end{aligned}$$

where the Weyl tensor is

$$C_{abcd} = R_{abcd} - \frac{2}{3} (g_{a[c} R_{d]b} - g_{b[c} R_{d]a}) + \frac{1}{6} g_{a[c} g_{d]b} R .$$

The first term is the mixed gauge-gravity anomaly (1.1). The second term includes a more conventional *Riemann*-squared term, which was expected to be in the same supermultiplet. The remaining terms are various matter terms and gravity-matter couplings. All of these are of the same order as the first two terms and so must be taken properly into account.

The complete four-derivative Lagrangian is clearly quite complicated and it would be overwhelming to solve the corresponding equations of motion by brute force. However, we have already determined most features of the solution using general principles. All that is needed is a single equation of motion to find the metric factor U . The simplest is to use the equation of motion for the auxiliary D field

$$\mathcal{N} = 1 - \frac{c_2 I}{72} (F_{ab}^I v^{ab} + M^I D) .$$

Inserting the expressions for the various fields in terms of charges and the metric factor we find

$$(4.7) \quad e^{-6U} = \frac{1}{6} c_{IJK} H^I H^J H^K + \frac{c_2 I}{24} (\nabla H^I \nabla U + 2 H^I \nabla^2 U) .$$

The two derivative theory corresponds to keeping only the first term on the right hand side and then the equation gives the metric factor U explicitly. In the four derivative theory the equation has become a second order differential equation for the metric factor U . This describes the deformation of the standard special geometry constraint (4.5) due to higher derivative corrections.

4.4. Attractor Solution. In the introduction we noted that, if the string solution is regular, the near string geometry will be $\text{AdS}_3 \times S^2$. The near string region is isolated by omitting the constant term in the harmonic functions

$$H^I = p^I / 2r ,$$

and a $\text{AdS}_3 \times S^2$ ansatz amounts to

$$e^{-6U} = \ell_S^3 / r^3 .$$

We have for definiteness written the *ansatz* in terms of the ℓ_S , the radius of the S^2 , but supersymmetry implies $U_1 = U_2$ which amounts to

$$\ell_A = 2\ell_S ,$$

so we could just as well have used ℓ_A , the radius of AdS_3 .

Now, inserting the *ansatz* into the deformed special geometry constraint (4.7) we see that the attractor *ansatz* is an exact solution with the identification of the ℓ_S as

$$(4.8) \quad \ell_S^3 = \frac{1}{8} \left(\frac{1}{6} c_{IJK} p^I p^J p^K + \frac{1}{12} c_{2I} p^I \right) .$$

The last term in this equation constitutes a correction to the scale ℓ_S . In the case of a string that is classically singular $c_{IJK} p^I p^J p^K = 0$ and so this term corrects the scale ℓ_S to a finite value. In this case the higher derivative terms have therefore resolved a singularity.

4.5. c-extremization. In two-derivative gravity the central charge and the scale of the AdS_3 space are related by the Brown-Henneaux formula (3.1). Indeed this was a source of intuition that the singularity should be resolved. At this point have seen how derivative corrections modify the spacetime solution but we have yet to compute the corrected central charge.

The total central charge $c = \frac{1}{2}(c_R + c_L)$ is the trace anomaly of the two-dimensional boundary CFT. The holographic manifestation of the trace anomaly is an anomalous variation of the on-shell action under rescaling of the boundary metric. This variation is essentially the on-shell action itself so the general central charge, including higher derivative corrections, equals the on-shell action of the attractor solution, up to a known overall numerical factor [12].

Evaluating the full action (4.4), (4.6) on the attractor solution determined in the previous subsection we find

$$(4.9) \quad c = -12\ell_A^3 \ell_S^2 (\mathcal{L}_0 + \mathcal{L}_1) = c_{IJK} p^I p^J p^K + \frac{3}{4} c_{2I} p^I ,$$

in agreement with the exact central charges (3.4) determined by anomaly inflow. More generally one can verify that the central charge (4.9) is extremized upon variation of the defining parameters of the attractor geometry. These agreements are sensitive consistency checks because the on-shell action depends on most terms in the actions (4.4), (4.6).

4.6. The interpolating solution. We have focussed so far on the behavior of the string solution in the important near string region. We now determine the complete radial evolution of the metric factor U in the special case of the solitonic heterotic string, represented as p $M5$ -branes wrapping the $K3$ of $CY_3 = K3 \times T^2$.

To do this we need to solve the deformed special geometry constraint (4.7). This is a nonlinear second order differential equation which cannot generally be solved analytically. However, we can proceed by solving one region at a time and then patch together partial results to determine the complete solution: first consider the attractor solution applicable at $r \sim p^{1/3}$ and extend it perturbatively to larger distances; and then consider the semiclassical solution around flat space, valid at $r \sim p$, and extend perturbatively towards smaller distances. Alternatively, we can simply find the numerical solution to the full differential equation. The results are presented in figure 1. It is seen that the expansion out from the attractor solution

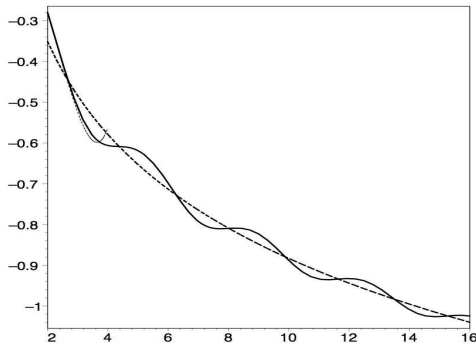


FIGURE 1. Metric factor interpolating between near string region and asymptotic space. *Solid line*: numerical solution. *Dashed line*: expansion around flat space. *Dotted line*: expansion around the near string solution.

matches up quite well with the expansion in from asymptotically flat space, with the numerical solution shadowing both.

The details of these procedures are given in [15] (and the review [18]). For the purpose of this talk the main point is simply that all this can be done, providing some confidence that the near horizon region attaches smoothly on to flat space in a rather conventional fashion.

4.7. Further corrections? We have included all corrections up to four derivative order in this work but one should ask whether there might further corrections? As mentioned in the introduction, this question is acute when corrections resolve a singularity for then the "corrections" are as important as the leading order terms. In other words, there is generally no systematic expansion parameter and so we must generally keep all orders in the Lagrangian.

In our explicit construction we have seen that the *form* of the solution is determined completely by supersymmetry. However, the final determination of the *scale* (4.8) requires solving the deformed special geometry constraint, an equation of motion which is expected to receive further corrections. Thus our approach is *not* generally immune to further corrections.

Our result (4.8) for the scale is an expansion with corrections controlled by an effective expansion parameter $1/p^2$. For large magnetic charge p , this structure precludes cancellation between different orders in the derivative expansion. However, for small black holes there is an additional challenge: some of the moduli are inherently small so higher order in inverse moduli can compensate for additional derivatives. For example, the dual heterotic string has strong coupling in the near string region, as one expects for a M5-brane. Therefore, it is not sufficient to consider solutions to the classical equations of motion as we do; quantum fluctuations around the saddle points may be significant.

The issue of quantum fluctuations can be circumvented by dualizing the type IIA solitonic string to the heterotic frame. The metric factor near the string becomes

$$e^{-6U} \rightarrow \ell^3/r^3, \quad \ell = \sqrt{\alpha'/2}.$$

The geometry of the heterotic string is thus $\text{AdS}_3 \times S^2$, with the AdS and sphere geometries of string scale. Note in particular that the scale of the geometry is independent of the number of fundamental strings. In this frame it is thus manifest that there is no systematic expansion parameter: the full geometry is stringy in nature. On the other hand, in the heterotic frame the dependence on the number of strings enters exclusively through the coupling constant which is fixed as

$$g_5^{\text{het}} = 2^{-1/4} p^{-1/2} ,$$

so quantum corrections are under good control for a large number of strings.

In the string geometry the precise definition of scale is ambiguous and one may try to maintain that central charge is a good proxy for scale. In that sense the higher derivative corrections resolve the gravitational singularity, but with other notions of length the resolution is merely qualitative, which are subject to further corrections of the same order as those that have been included.

5. The holographic dual of the heterotic string

The supergravity representation of the heterotic string is of string scale, but it is still reasonable to ask what the holographic dual is. The standard line of reasoning suggests that it should be a $D = 1+1$ CFT with $(8, 0)$ supersymmetry and R-symmetry at least $\text{SU}(2)$. The classical super-isometry group of the solution is $OSp(*|4)$ [24], so one might in fact expect that the dual theory represents an affine extension of that group. The challenge is that there exist no CFT with all these properties. There is simply too much supersymmetry.

A possible resolution to this conundrum is suggested by the existence of nonlinear superconformal algebras (NSCA's) with the correct symmetries [26]. The nonlinearity of these algebras refer to the supercurrent OPE which takes the form

$$G^i(z)G^j(0) \sim \frac{k}{\chi} \frac{\delta^{ij}}{z^3} + \frac{1}{\chi} \frac{J^a(T^a)^{ij}}{z^2} + \frac{2T\delta^{ij}}{z} + \frac{1}{2\chi} \frac{\partial_z J_z^a T^a}{z} + \frac{1}{2\chi k} \frac{J_z^a J_z^b P_{ab}^{ij}}{z} .$$

The last term is bilinear in the currents and so represents an unfamiliar nonlinearity. The nonlinear term depends on the tensor

$$P_{ab}^{ij} = \frac{1}{2} \{T^a, T^b\}^{ij} - 2\chi \delta_{ab} \delta^{ij} ,$$

where χ is a constant depending on the group. This tensor is such that the nonlinearities vanish for small groups like $\text{SU}(2)$ but not for larger R-symmetry groups. The nonlinear superconformal algebras [27, 28] are powerful but unfamiliar relatives to W-algebras. In the current setting we consider multi-string states so the suggestion is that NSCA's are important in string field theory.

An intriguing feature of the NSCA's is that they give nonlinear formulae for the central charge. For example, the conformal extension of $OSp(*|4)$ has central charge

$$(5.1) \quad c = -12p \left(1 + \frac{3}{2p}\right) .$$

The classical (large p) limit gives the Brown-Henneaux formula (except for the sign). However, $p \sim 1/g^2$ so the nonlinear algebra appears to identify nontrivial quantum corrections to the spacetime central charge. Interesting as this prospect may seem it cannot be emphasized enough that there are severe obstacles to this

optimistic interpretation. The biggest problem is that the relevant algebras apparently have no unitary representations. This problem is manifest in the expression (5.1) for the central charge, which is in fact negative. Unitarity violation is not only unacceptable on general grounds, it is also the wrong physics for a setting expected to be extremely stable because of the high degree of supersymmetry.

It is not clear what conclusion one should draw from this. On the one hand, the appearance of NCSAs is tantalizing, and almost inevitable given the symmetries of the situation. On the other hand, the most straightforward implementation clearly needs to be modified. This situation makes the challenge of finding a satisfying description of the theory even more interesting.

6. Discussion

In this talk we discussed the resolution of gravitational singularities with emphasis on the case of 5D string solutions with $\text{AdS}_3 \times S^2$ near string geometry. However, the techniques apply in many other cases where one seeks to resolve singularities, or to correct solutions that are already regular in the two-derivative theory. Some examples are:

- *Black holes in five dimensions* with $\text{AdS}_2 \times S^3$ near horizon geometry. The solutions are supported by electric charges so the Bianchi identity is trivial. Instead one must work out Gauss' law from the explicit action. Fortunately, it turns out that the result again reduces to a harmonic equation and the complete solution can be determined explicitly.
- *Rotating black holes* in five dimensions have less symmetry and are correspondingly more complicated. Although all terms in the four-derivative action (4.6) now contribute, Gauss' law remains integrable, and again the complete solution can be determined.
- *Rotating black holes on Taub-NUT base space* are more involved yet but, remarkably, all equations remain integrable in much the same way as the simpler case. The significance of Taub-NUT is that its "cigar"-form interpolates between a central R^4 region and asymptotic $R^3 \times S^1$. Thus these 5D solutions on Taub-NUT include the 4D black holes that have been considered previously. The detailed comparison reveals discrepancies which can be traced to subtleties in the definition of charge in the presence of higher derivative terms.

These results and others were worked out in [14, 15, 16] and were recently reviewed in detail in [18]. The combined body of examples verify previous indirect arguments by explicit computation, sort out discrepancies in those arguments, and uncovers numerous new results, including many still await interpretation. The explicit construction of supersymmetric solutions to five dimensional theories with higher derivatives has thus proven a fruitful endeavor.

Among many open problems, the most far-reaching involve the generalizations to other dimensions, including 10 and 11 dimensions, and theories with more general matter content. In such more general contexts there are often very different motivations for considering higher derivative corrections like, for example, the recent reports of the possible finiteness of $N = 8$ supergravity [29, 30]. Many of the higher derivative terms identified for different purposes could be employed also in

the context of the resolution of gravitational singularities. It is therefore reasonable to expect a fruitful interplay between different perspectives on higher derivative corrections to gravitational theories.

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References

- [1] K. Behrndt, G. Lopes Cardoso, B. de Wit, D. Lust, T. Mohaupt and W. A. Sabra, “Higher-order black-hole solutions in $N = 2$ supergravity and Calabi-Yau string backgrounds,” *Phys. Lett. B* **429**, 289 (1998) [arXiv:hep-th/9801081].
- [2] G. Lopes Cardoso, B. de Wit and T. Mohaupt, “Corrections to macroscopic supersymmetric black-hole entropy,” *Phys. Lett. B* **451**, 309 (1999) [arXiv:hep-th/9812082].
- [3] G. Lopes Cardoso, B. de Wit and T. Mohaupt, “Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes,” *Nucl. Phys. B* **567**, 87 (2000) [arXiv:hep-th/9906094].
- [4] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, “Stationary BPS solutions in $N = 2$ supergravity with R^2 interactions,” *JHEP* **0012**, 019 (2000) [arXiv:hep-th/0009234].
- [5] H. Ooguri, A. Strominger and C. Vafa, “Black hole attractors and the topological string,” *Phys. Rev. D* **70**, 106007 (2004) [arXiv:hep-th/0405146].
- [6] E. P. Verlinde, “Attractors and the holomorphic anomaly,” arXiv:hep-th/0412139.
- [7] A. Dabholkar, “Exact counting of black hole microstates,” *Phys. Rev. Lett.* **94**, 241301 (2005) [arXiv:hep-th/0409148].
- [8] A. Dabholkar, R. Kallosh and A. Maloney, “A stringy cloak for a classical singularity,” *JHEP* **0412**, 059 (2004) [arXiv:hep-th/0410076].
- [9] V. Hubeny, A. Maloney and M. Rangamani, “String-corrected black holes,” *JHEP* **0505**, 035 (2005) [arXiv:hep-th/0411272].
- [10] A. Sen, “How does a fundamental string stretch its horizon?,” *JHEP* **0505**, 059 (2005) [arXiv:hep-th/0411255].
- [11] A. Dabholkar, F. Denef, G. W. Moore and B. Pioline, “Exact and asymptotic degeneracies of small black holes,” *JHEP* **0508**, 021 (2005) [arXiv:hep-th/0502157]; “Precision counting of small black holes,” *JHEP* **0510**, 096 (2005) [arXiv:hep-th/0507014].
- [12] P. Kraus and F. Larsen, “Microscopic black hole entropy in theories with higher derivatives,” *JHEP* **0509**, 034 (2005) [arXiv:hep-th/0506176].
- [13] P. Kraus and F. Larsen, “Holographic gravitational anomalies,” *JHEP* **0601**, 022 (2006) [arXiv:hep-th/0508218].
- [14] A. Castro, J. L. Davis, P. Kraus and F. Larsen, “5D attractors with higher derivatives,” *JHEP* **0704**, 091 (2007) [arXiv:hep-th/0702072].
- [15] A. Castro, J. L. Davis, P. Kraus and F. Larsen, “5D Black Holes and Strings with Higher Derivatives,” *JHEP* **0706**, 007 (2007) [arXiv:hep-th/0703087].
- [16] A. Castro, J. L. Davis, P. Kraus and F. Larsen, “Precision entropy of spinning black holes,” *JHEP* **0709**, 003 (2007) [arXiv:0705.1847 [hep-th]].
- [17] K. Hanaki, K. Ohashi and Y. Tachikawa, “Supersymmetric Completion of an R^2 Term in Five-Dimensional Supergravity,” *Prog. Theor. Phys.* **117**, 533 (2007) [arXiv:hep-th/0611329].
- [18] A. Castro, J. L. Davis, P. Kraus and F. Larsen, “String Theory Effects on Five-Dimensional Black Hole Physics,” arXiv:0801.1863 [hep-th].
- [19] F. Larsen, arXiv:hep-th/0608191.
- [20] J. D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity,” *Commun. Math. Phys.* **104**, 207 (1986).
- [21] J. M. Maldacena, A. Strominger and E. Witten, “Black hole entropy in M-theory,” *JHEP* **9712**, 002 (1997) [arXiv:hep-th/9711053].

- [22] J. A. Harvey, R. Minasian and G. W. Moore, “Non-abelian tensor-multiplet anomalies,” JHEP **9809**, 004 (1998) [arXiv:hep-th/9808060].
- [23] P. Kraus, F. Larsen and A. Shah, “Fundamental Strings, Holography, and Nonlinear Superconformal Algebras,” arXiv:0708.1001 [hep-th].
- [24] J. M. Lapan, A. Simons and A. Strominger, “Nearing the Horizon of a Heterotic String,” arXiv:0708.0016 [hep-th].
- [25] A. Giveon and D. Kutasov, “Fundamental strings and black holes,” JHEP **0701**, 071 (2007) [arXiv:hep-th/0611062].
- [26] M. Henneaux, L. Maoz and A. Schwimmer, “Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity,” Annals Phys. **282**, 31 (2000) [arXiv:hep-th/9910013].
- [27] V. G. Knizhnik, “SUPERCONFORMAL ALGEBRAS IN TWO-DIMENSIONS,” Theor. Math. Phys. **66**, 68 (1986) [Teor. Mat. Fiz. **66**, 102 (1986)].
- [28] M. A. Bershadsky, “SUPERCONFORMAL ALGEBRAS IN TWO-DIMENSIONS WITH ARBITRARY N,” Phys. Lett. B **174**, 285 (1986).
- [29] M. B. Green, H. Ooguri and J. H. Schwarz, Phys. Rev. Lett. **99**, 041601 (2007) [arXiv:0704.0777 [hep-th]].
- [30] Z. Bern, J. J. Carrasco, D. Forde, H. Ita and H. Johansson, Phys. Rev. D **77**, 025010 (2008) [arXiv:0707.1035 [hep-th]].

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